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Electrical Resistivity of Dilute, Interacting Fermions

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A first-principles calculation of the initial decay of a current-carrying state is used to infer the electrical resistivity of *interacting* fermions (electrons or holes). This approach is useful when it is impractical to apply the Kubo formalism.

KEY WORDS: Electrical conductivity; electrical resistivity; transport quantum Boltzmann equation; quantum Fermi liquid.

1. INTRODUCTION

Sulewski *et al.*[1] noted that free carrier electrodynamics in the normal state ($T > T_c$) of the high-temperature superconductors are better parametrized by an anomalous scattering rate $\propto |\omega|$ near the Fermi surface (FS) than by the conventional ω^2 of Landau Fermi liquids[2] (LFL). Numerous other studies have found the dc electrical resistance in the normal phase of the high-temperature super conductors to be a linear function of the temperature T over a wide range of temperature.[3] One would like to understand the range of possibilities from a purely theoretical perspective.

Although Kubo's expression for the electrical *conductivity* is available (and it is exact in principle), in practice his formalism is quite cumbersome when two-body forces are strong (see review in ref. 4)[2]; the main problem is that the dc conductivity is *inversely* proportional to the strength of the scattering mechanism and thus has no straightforward expansion in powers

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[2] The first *formally* exact procedure for including many-body effects in the Kubo formulas for the Fermi liquid was due to Langer,[4] who later developed the Ward identities for impurity scattering.[5] The .Ward identities and their solution for the electron–phonon scattering mechanism in the same context were first obtained by Engelsberg and Schrieffer.[6] These works are reviewed in the book by Mahan.[7] $\mathrm{Im}\{1/\varepsilon\}$ is evaluated in his Eqs. (5.4.11) and related to conductivity in Eq. (7.1.8); Kubo formulas and the quantum Boltzmann equation are derived in his Chapter 7.

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of the concentration of the concentration of scatterers or their scattering lengths. The standard expression $\sigma = ne^2\tau/m^*(1 + i\omega\tau)$ suffices to show that, while at high frequencies one can dispense with an intimate knowledge of the scatterers, for $\omega\tau \ll 1$ one needs to compute the lifetime τ for momentum decay, which is generally different from the quasiparticle lifetime. If one seeks to use theory to interpret experiment, in order to reveal the source of the scattering mechanism, then is is not satisfactory just to "patch on" a collision lifetime.

But that is just one aspect of the problem. A second concern involves m^*. Is one required to include proper self-energy corrections in m^*, which in the CuO_2-based high-T_c superconductors might amount to a large, albeit unknown, correction? Moreover, while the electrical resistivities in various scattering channels are additive (Mathiessen's empirical rule), this is not true of the individual conductivities. Therefore when considering several such mechanisms it is more "natural" to calculate the *resistivity* of each.

A remedy might appear to lie in transport equations, such as the Boltzmann equation, with appropriate electron–impurity and electron-phonon collision integrals designed to yield the resistivities directly. But this simplification is illusory. The *classical* Boltzmann equation is *in principle invalid* in strongly interacting quantum systems, as it is based on an assumption that the conduction particles undergo ballistic trajectories between collisions. Still, *faute de mieux*, Boltzmann's equation *is* commonly used in high-T_c-related calculations (e.g., ref. 9) together with a (still controversial) assumption that the charge carriers in the two-dimensional CuO_2 planes are in one-to-one correspondence with free fermions.[3] Of course, the use of the ordinary Boltzmann equation is a staple in ordinary metals, where this assumption is not controversial.[11]

The question posed in the present paper is, can one resolve the problem from first principles without unnecessary approximations, but in a more physically intuitive way?

In partial resolution to this question, I shall infer the resistivity directly from the *decay* of a very *specially chosen* homogeneous current-carrying state. This decay can occur either through various types of impurity scattering or through the action of the ubiquitous electron–phonon interaction. It is not required to make any assumptions regarding the strength or nature of the two-body interactions, which could be the Coulomb interactions, or more simply, the pointlike (Hubbard) repulsive forces. However, for definiteness, we do evaluate the resulting formula

[3] For evidence supporting the existence of a Fermi surface, see, e.g., the survey by Pickett *et al.*[10]

approximately (using the RPA) and apply the results to various specific cases. Our results are then found to be in substantial agreement with other forms of transport theory,[4 9] although a discrepancy does appear as $T \to 0$ for purely phonon scattering: with our formalism one calculates a finite decay time for the current-carrying state *even at* $T = 0$, whereas conventional theory has the resistance *vanishing* as T^5. A future examination of this descrepancy may well shed some needed light on the ways in which strongly interacting systems shed their momentum.

In the present approach, we simply suppose a uniform charged particle density $\varkappa$ to be initially *in thermodynamic equilibrium in a moving coordinate system* of velocity $\mathbf{v}_0$ where it is parametrized by a given chemical potential μ and a temperature T. The particles are subject to the periodic potential of the solid as well as to their usual two-body forces *but not*, for $t \leqslant 0$, to any scattering mechanism capable of causing the momentum to decay. Thus, initially our system is a perfect conductor (although technically, it is not a *super*conductor.) The current is a constant of the motion for $t \leqslant 0$, and the current density is just $\mathbf{I} \equiv \varkappa \mathbf{v}_0$.

At $t = 0$, we "turn on" those scattering mechanisms (either impurities or phonons or both) which ultimately bring the current to a halt, and calculate the initial rate of decay of the current, $\partial \mathbf{I}/\partial t$, using the first Born approximation. This approximation is sufficiently accurate, if for simplicity, one assumes all the *momentum-nonconserving scattering* mechanisms to be *weak*. However, there is *no limitation* placed on the *momentum-conserving* two-body forces.

The result is found to be of the form $-\mathbf{I}/\tau$, where τ is independent of $\mathbf{v}_0$ in the limit $|\mathbf{v}_0| \to 0$. If there are several scattering mechanisms, each responsible for an inverse lifetime $1/\tau_n$, in the lowest Born approximation they are additive: $1/\tau = \sum_n 1/\tau_n$, in accord with Mathiessen's rule. (Of course, there can be interferences in higher order, as in the Kondo effect.) In lowest order, then, the total resistivity is $\rho = \sum_n \rho_n = m^*/\varkappa e^2 \tau$ with m^* just the *free particle* mass (i.e., the mass computed in the absence of two-body forces among the charge carriers).

Note that the many-body "quasiparticle mass" (with m^* determined in part by the proper self-energy corrections $\Sigma_{\mathbf{k}}(\omega)$ due to two-body interactions) does not appear in the present formalism. However, it will be necessary to know the one-body momentum distribution function $f(\mathbf{k}) \equiv \langle c^*_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} \rangle_{\mathrm{TA}}$ calculated in the presence of the two-body forces. But as $f(\mathbf{k})$ is, in fact, a function of $\Sigma_{\mathbf{k}}(\omega)$, the many-body aspects have not been altogether finessed!

To display our results in simplest form, we shall assume in this paper that the solution to the one-particle Schrödinger equation yields the energy of an individual fermion as $e(k) = \hbar^2 k^2/2m^*$, its momentum is $\hbar\mathbf{k}$, and its

velocity is $\mathbf{v} = \hbar\mathbf{k}/m^*$, and m^* is a (scalar) constant. Generalization to anisotropic masses or to arbitrary band structures is straightforward (even though it is not trivial) and is omitted.

The one-body Hamiltonian for a larger number of particles is

$$H_0 = \sum_{\mathbf{k},\sigma} [e(k) - \mu]\, c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} = \sum_{\mathbf{k},\sigma} [e(k) - \mu]\, n_{\mathbf{k}\sigma} \tag{1}$$

The total momentum operator is

$$\mathbf{P}_{\text{op}} = \sum_{\mathbf{k},\sigma} \hbar\mathbf{k} c^*_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} = \sum_{\mathbf{k},\sigma} \hbar\mathbf{k} n_{\mathbf{k}\sigma} \tag{2}$$

and, by the above, it is proportional to the particle-current operator; two-body forces play no role in the definitions of these operators.

In units such that the lattice parameter $a \equiv 1$, the particle-current density initially is $\mathbf{I}$:

$$\mathbf{I} = N^{-1} \sum_{\mathbf{k},\sigma} (\hbar\mathbf{k}/m^*) \langle \mathbf{n}_{\mathbf{k}-\mathbf{k}_0\sigma} \rangle_{\text{TA}} = \mathbf{v}_0 N^{-1} \sum_{\mathbf{k},\sigma} \langle n_{\mathbf{k}\sigma} \rangle_{\text{TA}} = \mathbf{v}_0 n \tag{3}$$

where $\hbar\mathbf{k}_0 = m^*\mathbf{v}_0$. The subscript TA stands for "thermal average" in an ensemble described by $\mathscr{H} \equiv H_0 + \lambda H_2$, where H_2 contains the two-body interactions and λ is a coupling parameter set equal to 0 or 1. The identity $\mathbf{I} = \mathbf{v}_0 n$ demonstrates the proportionality of two initially conserved quantities, particle number and current.

2. THE DERIVATION

We assume the eigenstates of $\mathscr{H} = H_0$ (the kinetic energy) $+ \lambda H_2$ (momentum-conserving two-body interactions, e.g., Coulomb repulsion) are known. The Hamiltonian $\mathscr{H}$ governs the dynamics of what is assumed to be a *low-density nonmagnetic fluid of fermions.* In such low-density systems the Coulomb interactions are typically quite significant. But two-body forces depend only on the separation of an interacting pair, and thus conserve momentum to within a reciprocal lattice vector $\hbar\mathbf{K}_n$. Thus, at $t < 0$, prior to introduction of some specific scattering mechanisms, the eigenstates of $\mathscr{H}$ can also be chosen to be eigenstates of $\mathbf{P}_{\text{op}}$ in (2) with eigenvalues $\mathbf{P}$.

There is no need to worry about "Umklapp," the discontinuous change in $\mathbf{P}$ by some integer multiple of a reciprocal lattice vector $\hbar\mathbf{K}_n$ (where the lost momentum is given up to the massive lattice.) This process is akin to diffraction, and requires that k_F be $O(\pi/a)$. But at low densities $k_F \ll \pi/a$; therefore Umklapp is an essentially negligible mechanism for

current decay via the two-body collisions. This is one reason we limit the discussion to a low-density fluid; the other is the applicability of the effective-mass approximation. With these assumptions momentum and particle current are conserved and proportional to each other. This allows us to label exact eigenstates $|\alpha, \mathbf{P}\rangle$ of $\mathscr{H}$ by two quantum numbers: the energy E_α and the momentum $\mathbf{P}$.

The non-momentum-conserving perturbations H' will now be tacked on and treated in lowest-order perturbation theory. These cause current to decay while, on average, preserving the homogeneity of the density of carriers. The scatterers either are nonmagnetic impurities H_I' randomly distributed over N_I sites at R_j $(j=1, ..., N)$,

$$H_I' \equiv N^{-1} \sum_{\mathbf{q}} \sum_{j} e^{i\mathbf{q}\cdot\mathbf{R}_j} V(\mathbf{q}) \sum_{\mathbf{k},\sigma} c^*_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma} \tag{4}$$

or they are randomly distributed magnetic impurities,

$$H_1' \equiv N^{-1} \sum_{\mathbf{q}} \sum_{j} e^{i\mathbf{q}\cdot\mathbf{R}_j} \mathbf{U}(\mathbf{q}) \sum_{k,\sigma,\sigma'} \mathscr{M}(\sigma,\sigma')\, c^*_{\mathbf{k}+\mathbf{q},\sigma'} c_{\mathbf{k},\sigma} \tag{5}$$

where $\mathscr{M}$ is one of 3 Pauli matrices. And finally, there are always the phonons:

$$H'_{\mathrm{e-ph}} \equiv \sum_{\mathbf{q}} \mathscr{V}(\mathbf{q})(a_{\mathbf{q}} + a^*_{-\mathbf{q}})(\hbar\omega_{\mathbf{q}}/\hbar\omega_D N)^{1/2} \sum_{k,\sigma} c^*_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma} \tag{6}$$

governed by a Hamiltonian which, in the Debye model, is simply

$$H_{\mathrm{ph}} = \sum_{\mathbf{q}} \hbar\omega_q a^*_{\mathbf{q}} a_{\mathbf{q}}, \qquad \text{with} \quad \omega_q \approx sq \quad \text{for} \quad q < \pi \quad \text{and} \quad \omega_\pi \equiv \omega_{\mathrm{D}} \tag{7}$$

As the couplings $V(\mathbf{q})$, $\mathbf{U}(\mathbf{q})$, or $\mathscr{V}(\mathbf{q})$ are assumed weak, each individual scattering mechanism can be examined separately. And because there is no interference among channels in lowest Born approximation, the scattering rates are additive in agreement with Mathiessen's empirical rule. Kondo-type phenomena arising from higher-order interferences among distinct scattering mechanisms could ultimately can be examined by extending the present formalism to the *second* Born approximation.

3. NONMAGNETIC IMPURITY SCATTERING

The texbook formula for the rate of decay w of initial states $|\alpha\rangle$ into final states $|\beta\rangle$ in the first Born approximation, often referred to as "Fermi's golden rule," is $w = 2\pi/\hbar \sum_\beta \langle |\mathrm{M.E.}_{\alpha\to\beta}|^2 \rangle_{\mathrm{TA}}\, \delta(E_\alpha - E_\beta)$, where

M.E.$_{\alpha \to \beta}$ stands for the corresponding matrix element of a current-nonconserving perturbation H'. One averages over initial states (α) and sums over final states (β).

In lowest Born approximation each type of momentum transfer can be considered a separate channel and treated individually. The rate of decay of the current density via momentum transfer $\hbar\mathbf{q}$ is governed by the operator

$$H'(\mathbf{q}) \equiv [V(\mathbf{q})/N] \sum_j \rho(\mathbf{q})\, e^{i\mathbf{q}\cdot\mathbf{R}_j} \tag{8}$$

in which the density-fluctuation operator $\rho(\mathbf{q}) \equiv \sum_{k,\sigma} c^*_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma}$ is used for typographic simplicity. Then, the rate of decay of an average current-carrying state by loss of a designated momentum $\hbar\mathbf{q}$ is

$$\begin{aligned} w(\mathbf{q}) &\equiv 2\pi/\hbar \sum_{\alpha,\beta} e^{-\beta E_\alpha} |\langle\beta|\, H'(\mathbf{q})\, |\alpha\rangle|^2\, \delta(E_\alpha - E_\beta)/Z \\ &= 2\pi N_I/\hbar \sum_{\alpha,\beta} e^{-\beta E_\alpha} |(V(\mathbf{q})/N)\, \langle\beta|\, \rho(\mathbf{q})\, |\alpha\rangle|^2\, \delta(E_\alpha - E_\beta)/Z \\ &\quad + \left\{ 4\pi/\hbar \sum_\beta \sum_i \sum_{j>i} e^{i\mathbf{q}\cdot\mathbf{R}_{ij}} (V(\mathbf{q})/N)^2 \cdots \right\} \end{aligned}$$

where $Z = \sum \exp -\beta E_\alpha$ is the partition function. If the impurities are seeded at random, as we shall assume, then two-center terms $\{\cdots e^{i\mathbf{q}\cdot\mathbf{R}_{ij}} \cdots\}$ all vanish upon averaging over positions, regardless of the strength of the scattering mechanism. As a result, the rate of total momentum decay in a specified momentum channel, $\partial\mathbf{P}/\partial t|_{\mathbf{q}}$, is

$$\partial\mathbf{P}/\partial t|_{\mathbf{q}} = 2\pi\, |V(\mathbf{q})|^2\, (N_I/N^2)\, \mathbf{q} \sum_{\alpha,\beta} e^{-\beta E_\alpha} |\langle\beta|\, \rho(\mathbf{q})\, |\alpha\rangle|^2\, \delta(E_\alpha - E_\beta)/Z \tag{9}$$

In thermal equilibrium, the double sum is proportional to $\lim_{\omega\to 0} \{kT\omega^{-1}\, \mathrm{Im}(\varepsilon^{-1}(\mathbf{q},\omega)\}$.[7] This is our general result.

Unfortunately, this formula is in the form of a singular limit which cannot be obtained directly. What is more, according to our chosen boundary condition, the system is initially in thermal equilibrium in a *moving* coordinate system. $\varepsilon(\mathbf{q},\omega)$ is affected by the motion to $O(\mathbf{v}_0)$, and we need to know the corrections precisely to that order.

To illustrate the procedure we first compute the trivial case of *free fermions*, which can be obtained directly. This provides a benchmark for the interacting case. Setting $\lambda = 0$ in $\mathscr{H}$, we find that (9) yields straightaway

$$\partial \mathbf{P}/\partial t|_{\mathbf{q}} = 4\pi |V(\mathbf{q})|^2 N_I \times N^{-2} \sum_k \sum_{k'} (\mathbf{k}' - \mathbf{k})\, \delta(\mathbf{q} - \mathbf{k}' + \mathbf{k})\, f_0(\mathbf{k} - \mathbf{k}_0) \times [1 - f_0(\mathbf{k}' - \mathbf{k}_0)]\, \delta(e(\mathbf{k}) - e(\mathbf{k}')) \tag{10}$$

where $f_0(e(\mathbf{k}))$, abbreviated $f_0(\mathbf{k}) = [1 + \exp(e(\mathbf{k}) - \mu)/kT]^{-1}$, is the Fermi–Dirac distribution function for free fermions of energy $e(\mathbf{k})$. As the carriers are initially in thermal equilibrium in the *moving* coordinate system, the $\mathbf{k}$'s in the f_0's in Eq. (10) are referred to an origin at $\mathbf{k}_0$. A factor 2 is included for $\sum_\sigma$. Summing over $\mathbf{q}$ to find the *total* rate of decay, and shifting the origin of $\mathbf{k}$, $\mathbf{k}'$ to $\mathbf{k}_0$, we obtain

$$\begin{aligned}\partial \mathbf{P}/\partial t &= 2\pi N_I N^{-2} \sum_k \sum_{k'} \sum_{\mathbf{q}} \mathbf{q}\, |V(\mathbf{q})|^2\, \delta(\mathbf{q} - \mathbf{k}' + \mathbf{k})\, \delta(e(\mathbf{k}) - e(\mathbf{k}') - \hbar \mathbf{q} \cdot \mathbf{v}_0) \\ &\quad \times \{f_0(\mathbf{k})[1 - f_0(\mathbf{k}')] - f_0(\mathbf{k}')[1 - f_0(\mathbf{k})]\} \\ &= 2\pi N_I N^{-2} \sum_k \sum_{\mathbf{q}} \mathbf{q}\, |V(\mathbf{q})|^2\, \delta(e(\mathbf{k}) - e(\mathbf{k} + \mathbf{q}) - \hbar \mathbf{q} \cdot \mathbf{v}_0) \\ &\quad \times [f_0(\mathbf{k}) - f_0(\mathbf{k} + \mathbf{q})] \end{aligned} \tag{11}$$

With $\mathcal{N}(e)$ the 3D one-particle density of states $= (2\pi)^{-2} (2m^* a^2/\hbar^2)^{3/2} \sqrt{e}$ and $\hbar^2 k_F^2/2m^* = \mu$, one proceeds to the limit $|\mathbf{v}_0| \to 0$ and obtains

$$\partial \mathbf{P}/\partial t = 2\pi N_I N^{-2} \sum_k \sum_q \mathbf{q}\, |V(\mathbf{q})|^2\, \hbar \mathbf{q} \cdot \mathbf{v}_0 [\partial f_0(\mathbf{k})/\partial e(\mathbf{k})]\, \delta(e(\mathbf{k}) - e(\mathbf{k} + \mathbf{q})) \tag{12a}$$

$$= -\hbar \mathbf{v}_0 (6\pi a)^{-1} N_I \left[N^{-1} \sum_q q\, |V(\mathbf{q})|^2\, (m^* a^2/\hbar^2)^2\, f_0(\hbar^2 q^2/8m^*) \right] \tag{12a'}$$

upon reintroducing the lattice parameter a to fix the units: $\hbar^2/m^* a^2 \approx O(1/6) \times$ electron bandwidth. Thus, *the initial rate of decay of the current is exponential* and characterized by an inverse lifetime $1/\tau_0 = -\partial \mathbf{P}/\partial t \div \mathbf{P}$:

$$1/\tau_0 = \hbar (6\pi a \varkappa m^*)^{-1} c_I \left[N^{-1} \sum_{\mathbf{q}} q\, |V(\mathbf{q})|^2\, (m^* a^2/\hbar^2)^2\, f_0(\hbar^2 q^2/8m^*) \right] \tag{13a}$$

where $c_I \equiv N_I/N$ and $\varkappa = N_{el}/N$. (Note that the density-of-states factor results in $1/\tau_0 \propto m^*$, a proportionality which is not usually highlighted in standard studies.) The corresponding resistivity is

$$\rho_0 = \hbar (6\pi a \varkappa^2 e^2)^{-1} c_I \left[N^{-1} \sum_{\mathbf{q}} q\, |V(\mathbf{q})|^2\, (m^* a^2/\hbar^2)^2\, f_0(\hbar^2 q^2/8m^*) \right] \tag{14a}$$

where e is the charge of the carriers. If $|V(\mathbf{q})|^2$ is assumed constant, as is reasonable for a short-range scattering center, for $kT \ll \mu$ we have $f_0 \approx 1$ for $q < 2k_F$ and ≈ 0 for $q > 2k_F$. Then the above bracket $[\cdots] \propto k_F^2 \propto n^{4/3}$, and $\rho_0 \propto m^{*2} n^{-2/3}$ for $n \ll 1$—*not* $\rho_0 \propto m^* n^{-1}$ as naively expected!

The corresponding formulas in 2D are only marginally different: Eq. (12a′) is replaced by

$$\partial \mathbf{P}/\partial t = -\hbar v_0 (4\pi a^2)^{-1} N_I \left[N^{-1} \sum_{\mathbf{q}} q\, |V(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \operatorname{Re}(k_F^2 - (q/2)^2)^{1/2} \right] \tag{12b}$$

where $\operatorname{Re}(X)$ stands for the real part of (X) and again we assume $kT \ll \mu$.

In 2D, Eq. (13a) is replaced by

$$1/\tau_0 = \hbar (4\pi a^2)^{-1} (n m^*)^{-1} c_I \times \left[N^{-1} \sum_{\mathbf{q}} q\, |V(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \operatorname{Re}(k_F^2 - (q/2)^2)^{-1/2} \right] \tag{13b}$$

and (14a) by

$$\rho_0 = \hbar (4\pi a^2)^{-1} (n^2 e^2)^{-1} c_I \times \left[N^{-1} \sum_{\mathbf{q}} q\, |V(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \operatorname{Re}(k_F^2 - (q/2)^2)^{-1/2} \right] \tag{14b}$$

If $|V(\mathbf{q})|^2$ is again assumed constant, then the bracket $[\cdots] \propto k_F^2 \propto n$. Thus, in 2D, $\rho_0 \propto n^{-1}$ for $n \ll 1$, the dependence on n (but not on m^*) now being *precisely* as naively expected!

Now, turn to *interacting fermions* ($\lambda = 1$). For real values of x [such as $\hbar^{-1}(E_\alpha - E_\beta)$], it is convenient to use $\delta(x) = \delta(-x) = (1/2\pi) \int_{-\infty}^{\infty} dt \exp ixt$ together with retarded or advanced Green functions in the evaluation of thermodynamic averages.[12] As a first step, introduce the time-dependent operators $\rho(\mathbf{q}, t) = e^{i\mathscr{H}t} \rho(\mathbf{q}) e^{-i\mathscr{H}t}$ to simplify $\partial \mathbf{P}/\partial t$:

$$\partial \mathbf{P}/\partial t = 2\pi\hbar^{-1} \sum_{\mathbf{q}} |V(\mathbf{q})|^2 (N_I/N) \mathbf{q} \left\{ (1/2\pi) \int_{-\infty}^{\infty} dt\, N^{-1} \langle \rho(\mathbf{q}, t)\, \rho(-\mathbf{q}) \rangle_{TA} \right\} \tag{15}$$

The correlation function $N^{-1} \langle \rho(\mathbf{q}, t)\, \rho(-\mathbf{q}) \rangle_{TA}$ is computed with the aid of the corresponding retarded Green function $G(\mathbf{q}, \omega)$:

$$N^{-1} \langle \rho(\mathbf{q}, t)\, \rho(-\mathbf{q}) \rangle_{TA} = (2\pi i)^{-1} \int d\omega\, e^{i\omega t} [e^{\beta\omega} - 1]^{-1} \{ G(\mathbf{q}, \omega - i0^+) - G(\mathbf{q}, \omega + i0^+) \} \tag{16}$$

We make use of Eq. (16) by first solving the equations of motion of $G(\mathbf{q}, \omega)$ (using standard RPA decoupling):

$$G(\mathbf{q}, \omega) \approx \hbar N^{-1} \sum_{\mathbf{k},\sigma} \frac{f(\mathbf{k}+\mathbf{q}-\mathbf{k}_0) - f(\mathbf{k}-\mathbf{k}_0)}{\hbar\omega - e(\mathbf{k}) + e(\mathbf{k}+\mathbf{q})} \times \varepsilon^{-1}(\mathbf{q}, \omega)$$

$$\equiv \hbar \Pi(\mathbf{q}, \omega) \times \varepsilon^{-1}(\mathbf{q}, \omega) \tag{17}$$

The above serves to define the polarization function $\Pi(\mathbf{q}, \omega)$. Both the dielectric function $\varepsilon(\mathbf{q}, \omega)$ and $\Pi(\mathbf{q}, \omega)$ are to be evaluated in the moving system, with ε being related to Π in the usual way:

$$\varepsilon(\mathbf{q}, \omega) = 1 - U_{\mathbf{q}} \Pi(\mathbf{q}, \omega) \tag{18}$$

where $U_{\mathbf{q}}$ is the Fourier transform of the two-body potential and is a *real* variable.[4] Equation (17) serves to define Π; we emphasize that both Π and ε are functions of $\mathbf{k}_0$. Insertion of (17) into (16) yields the correlation function:

$$N^{-1}\langle \rho(\mathbf{q}, t)\, \rho(-\mathbf{q}) \rangle_{\mathrm{TA}}$$

$$= (\hbar / U_{\mathbf{q}} \pi) \int_{-\infty}^{+\infty} d\omega\, e^{i\omega t} \frac{1}{e^{\beta\omega} - 1} \operatorname{Im}\{\varepsilon^{-1}(\mathbf{q}, \omega)\}$$

$$= 2\hbar \int_{-\infty}^{+\infty} d\omega\, e^{i\omega t} \frac{1}{e^{\beta\omega} - 1} \frac{1}{N}$$

$$\times \sum_{\mathbf{k}} \left\{ \frac{[f(\mathbf{k}+\mathbf{q}-\mathbf{k}_0) - f(\mathbf{k}-\mathbf{k}_0)]\, \delta(\hbar\omega - e(\mathbf{k}) + e(\mathbf{k}+\mathbf{q}))}{|\varepsilon(\mathbf{q}, \omega)|^2} \right\} \tag{19}$$

Formally, this is similar to free fermions except that $f(\mathbf{k})$ replaces $f_0(\mathbf{k})$ and $|\varepsilon(\mathbf{q}, \omega)|^2$ appears in the denominator. In evaluating (15), we use $1/2\pi \int dt\, e^{i\omega t}$ in its symmetric form $1/2 \times (\delta(\omega) + \delta(-\omega))$. This requires that ω be everywhere replaced by 0—except in the singular term $[e^{\beta\omega} - 1]^{-1}$, which must be replaced by its symmetric part, $-1/2$, before proceeding to the limit. After a shift of origin of $\mathbf{k}$ by $\mathbf{k}_0$, the energies $e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})$ are changed to $e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q}) - \hbar\mathbf{q} \cdot \mathbf{v}_0$ as before. So, finally, upon proceeding to the limit $\mathbf{v}_0 \to 0$,

$$\partial \mathbf{P} / \partial t = 2\pi N_{\mathrm{I}} N^{-2} \sum_{k} \sum_{k'} \sum_{\mathbf{q}} \mathbf{q}\, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2$$

$$\times \delta(\mathbf{q} - \mathbf{k}' + \mathbf{k})\, \delta(e(\mathbf{k}) - e(\mathbf{k}') - \hbar\mathbf{q} \cdot \mathbf{v}_0)[f(\mathbf{k}) - f(\mathbf{k}')]$$

$$= 2\pi N_{\mathrm{I}} N^{-2} \sum_{k} \sum_{\mathbf{q}} \mathbf{q}\, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2 \hbar\mathbf{q}$$

$$\cdot \mathbf{v}_0 (\partial f(\mathbf{k}) / \partial e(\mathbf{k}))\, \delta(e(\mathbf{k}) - e(\mathbf{k}+\mathbf{q})) \tag{20}$$

[4] In the Hubbard model, $U_{\mathbf{q}}$ is independent of $\mathbf{q}$; for the Coulomb gas with $q \ll \pi/a$, $U_{\mathbf{q}} \approx 4\pi e^2/q^2 a^3$ in a 3D solid, and $U_{\mathbf{q}} \approx 2\pi e^2/qa^2$ in 2D.

Except for two "minor" details, this expression agrees with that for free fermions, Eq. (12a). The differences brought about by the two-body forces are: (a) the static screening of the scattering potential (as first intuited by Friedel[13]) by $\varepsilon^{-1}(\mathbf{q}) \equiv \varepsilon^{-1}(\mathbf{q}, 0)$, which transforms a *charged* scattering center into a screened short-ranged potential or an *un*charged scatterer into a much weaker one, and (b) replacement of $\partial f_0(e(\mathbf{k}))/\partial e(\mathbf{k}) \approx -\delta(e(\mathbf{k}) - \mu)$ by $\partial f(e(\mathbf{k}))/\partial e(\mathbf{k})$, which is less strongly singular at μ. When the two-body forces are strong, $\partial f(e(\mathbf{k}))/\partial e(\mathbf{k})$ does not vanish away from the Fermi surface and, even at low T, differs considerably from the psuedo-delta-function $\partial f_p(e(\mathbf{k}))/\partial e(\mathbf{k})$.[14]

After a partial integration, we have

$$1/\tau = h(6\pi a \varkappa m^*)^{-1} c_1 \left[N^{-1} \sum_{\mathbf{q}} q \, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2 \, (m^* a^2/h^2)^2 \, f(h^2 q^2/8m^*) \right] \tag{21a}$$

in 3D. The resulting resistivity is

$$\tau = h(6\pi a \varkappa^2 e^2)^{-1} c_1 \left[N^{-1} \sum_{\mathbf{q}} q \, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2 \, (m^* a^2/h^2)^2 \, f(h^2 q^2/8m^*) \right] \tag{22a}$$

The similarity of Eq. (22a) with (14a) is striking.

The expressions are again somewhat more cumbersome in 2D. Replacing (13b), we find

$$1/\tau = h(4\pi a^2)^{-1} (\varkappa m^*)^{-1} c_1 \left[N^{-1} \sum_{\mathbf{q}} q \, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2 \, (m^* a^2/h^2)^2 \right.$$
$$\left. \times \int dk \, |\partial f/\partial k| \, \mathrm{Re}(k^2 - (q/2)^2)^{-1/2} \right] \tag{21b}$$

and replacing (14b), a resistivity

$$\rho = h(4\pi a^2)^{-1} (\varkappa e^2)^{-1} c_1 \left[N^{-1} \sum_{\mathbf{q}} q \, |V(\mathbf{q})/\varepsilon(\mathbf{q})|^2 \, (m^* a^2/h^2)^2 \right.$$
$$\left. \times \int dk \, |\partial f/\partial k| \, \mathrm{Re}(k^2 - (q/2)^2)^{-1/2} \right] \tag{22b}$$

In closing this section, we note that $\Pi(\mathbf{q}, 0)$ is negative, thus $\varepsilon(\mathbf{q})$ is typically > 1, even in some instances $\gg 1$.

4. MAGNETIC IMPURITY SCATTERING

Screening of magnetic impurities occurs via magnetic, not charge, fluctuations. Hence $\varepsilon(\mathbf{q}, \omega) = 1 - U_q \Pi(\mathbf{q}, \omega)$ in the above formulas is replaced by $\varepsilon_{\text{mag}}(\mathbf{q}, \omega) = 1 + U_{\text{ex}} \Pi(\mathbf{q}, \omega)$, with U_{ex} measuring the strength of the exchange forces and $1/\varepsilon_{\text{mag}}$ the paramagnetic *enhancement* (i.e., *anti*-screening!) factor.[14] Unlike the dielectric function, ε_{mag} is typically < 1, and can even vanish on discrete points of surfaces signaling a magnetic instability for strongly interacting fermions. For the charged gas, the parameter $U_{\text{ex}} = O(e^2/a)$ and, unlike U_q, it is approximately constant over the Brillouin zone, reflecting the short range of exchange forces in real space. In Hubbard's model, U and U_{ex} are identical parameters.

Thus, Eqs. (21)–(22) are immediately applicable for magnetic impurities, upon replacement of $|V(\mathbf{q})/\varepsilon(\mathbf{q})|^2$ by $|\mathbf{U}(\mathbf{q})/\varepsilon_{\text{mag}}(\mathbf{q})|^2$. Like U_{ex}, the magnetic impurity's scattering matrix element $\mathbf{U}(\mathbf{q})$ also tends to be approximately independent of q.

For dilute fluids, this substitution has important consequences. If U_{ex} is large compared with μ, ε_{mag} will be small either near $q = 0$ or $2k_F$, and the resulting enhancement of scattering cross section for the magnetic impurities can be arbitrarily great. Such divergence in 3D signals emergence of a magnetic symmetry-broken phase. In 2D, however, Chen and Mattis[14] have shown the new phase to be paramagnetic and without long-range order. This phase they denoted the "quantum Fermi liquid" (QFL). The QFL exists only above a critical value of $U_{\text{ex}} \approx 17\mu$. In this phase, the resistivity due to *magnetic* impurities is predicted to be anomalously large as compared to that of nonmagnetic impurities.

5. SCATTERING BY PHOTONS

We now invoke the electron–phonon matrix element in (6). Equations (11) and (20) are now replaced by

$$\partial \mathbf{P}/\partial t = 4\pi N^{-1} \sum_k \sum_{k'} \sum_{\mathbf{q}} \mathbf{q} \, |\mathscr{V}(\mathbf{q})(\hbar\omega_{\mathbf{q}}/\hbar\omega_{\text{D}})^{1/2} \varepsilon^{-1}(\mathbf{q})|^2 \, \delta(\mathbf{q} - \mathbf{k}' + \mathbf{k}) \times [f(\mathbf{k}) - f(\mathbf{k}')] \{\delta(\hbar\omega_q + e(\mathbf{k}) - e(\mathbf{k}') - \hbar\mathbf{q} \cdot \mathbf{v}_0)[2n(\hbar\omega_{\mathbf{q}}) + 1]\} \tag{23}$$

which we shall once again evaluate in the limit $|\mathbf{v}_0| \to 0$. The phonon distribution function is the usual Bose–Einstein function, $n(\hbar\omega_{\mathbf{q}}) =$

$[e^{\beta\hbar\omega_q}-1]^{-1}$. Then, in 3D in the so-called "adiabatic" limit (in which the ratio of speed of sound to Fermi velocity vanishes, $s/v_F \to 0$)

$$\partial \mathbf{P}/\partial t = -\hbar \mathbf{v}_0 (6\pi a)^{-1} N^{-1} \sum_{\mathbf{q}} q \, |\hat{\mathscr{V}}(q)(\omega_q/\omega_D)^{1/2} \varepsilon^{-1}(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \times \coth(\beta\hbar\omega_q/2) \, f(\hbar^2 q^2/8m^*) \tag{24}$$

hence

$$1/\tau_{\text{e-ph}} = \hbar(6\pi\nu m^* a)^{-1} N^{-1} \sum_{\mathbf{q}} q \, |\hat{\mathscr{V}}(q)(\omega_q/\omega_D)^{1/2} \varepsilon^{-1}(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \times \coth(\beta\hbar\omega_q/2) \, f(\hbar^2 q^2/8m^*) \tag{25a}$$

and the resistivity is

$$1/\tau_{\text{e-ph}} = \hbar(6\pi\nu^2 e^2 a)^{-1} N^{-1} \sum_{\mathbf{q}} q \, |\hat{\mathscr{V}}(q)(\omega_q/\omega_D)^{1/2} \varepsilon^{-1}(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \times \coth(\beta\hbar\omega_q/2) \, f(\hbar^2 q^2/8m^*) \tag{26a}$$

Consider the "high-temperature" regime in which $\hbar\omega_q < kT$ for all $q \leqslant 2k_F$. For $\nu = 1$ the high-temperature regime is defined by $T > \theta_D$, where the Debye temperature is $\theta_D \equiv \hbar\omega_D/k_B$. If $\hat{\mathscr{V}}/\varepsilon \approx$ const, the integration over $\mathbf{q}$ is performed as follows: defining $\beta\hbar s/2 \equiv q_T^{-1}$, the integral is (crudely) of the form

$$I = \int_0^{2k_F} dq \, q^4 \coth q/q_T = q_T^5 \int_0^{2k_F/q_T} dx \, x^4 \coth x \tag{27}$$

At high temperature, where $2k_F \ll q_T$, $I \approx \frac{1}{4} q_T^5 (2k_F/q_T)^4 \propto k_F^4 \times T$. Thus the high-temperature resistivity associated with the electron–phonon initeraction is proportional to $Tm^{*2}\nu^{-2/3}$.

At $T = 0$, $q_T \to 0$ and $I = 1/5(2k_F)^5$. As a result, the $T = 0$ resistivity is proportional to k_F^{-1}, i.e., is inversely proportional to $\nu^{1/3}$. But aside from its dependence on ν, this residual resistivity *is finite* and cannot be distinguished from the contribution of impurities. (This result contrasts with the conventional transport theory, which predicts that the low-T electrical resistivity will vanish as T^5 at absolute zero in pure metals devoid of scattering centers.) To obtain the temperature dependence of the phonon resistivity at low temperatures within the present theory, one subtracts the $T = 0$ contribution from the total, to obtain a quantity proportional to

$$I - I_0 = \int_0^{2k_F} dq \, q^4 (\coth q/q_T - 1) \propto q_T^5 \tag{28}$$

which recovers the familiar result $\rho - \rho_0 \propto T^5$.

The crossover from the low-temperature $\rho \propto T^5$ to the high-temperature $\rho \propto T$ behavior has been shown to occur at $T_x \approx \frac{1}{4}\theta_D$ for a wide range of ordinary metals in which $\varkappa = O(1)$.[15] It follows that for $\varkappa \ll 1$, the crossover will be at $T_x \approx \frac{1}{4}\theta_D \varkappa^{1/3}$.

The corresponding expressions in 2D are

$$1/\tau_{e\text{-ph}} = \hbar(4\pi a^2)^{-1}(\varkappa m^*)^{-1} \times \left[N^{-1} \sum_{\mathbf{q}} q\, |\hat{\mathcal{V}}(q)(\omega_q/\omega_D)^{1/2}\, \varepsilon^{-1}(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \coth(\beta\hbar\omega_q/2) \times \int dk\, |\partial f/\partial k|\, \mathrm{Re}(k^2 - (q/2)^2)^{-1/2} \right] \tag{25b}$$

and

$$\rho_{e\text{-ph}} = \hbar(4\pi a^2)^{-1}(\varkappa^2 e^2)^{-1} \times \left[N^{-1} \sum_{\mathbf{q}} q\, |\hat{\mathcal{V}}(q)(\omega_q/\omega_D)^{1/2}\, \varepsilon^{-1}(\mathbf{q})|^2 (m^* a^2/\hbar^2)^2 \coth(\beta\hbar\omega_q/2) \times \int dk\, |\partial f/\partial k|\, \mathrm{Re}(k^2 - (q/2)^2)^{-1/2} \right] \tag{26b}$$

Thus, one estimates the high-T resistivity in 2D to be $\propto Tm^{*2}/n$.

It is also possible to derive the resistivity of 2D electrons interacting with 3D phonons, but the resulting expression, which includes elliptic functions, is rather involved and is omitted here for brevity.

Our formulas in the limit $T = 0$ indicate that the Fermi fluid loses its momentum at a finite rate owing to the zero-point fluctuations in the phonons. Conversely, conventional transport theory predicts $\tau_{el\text{-}ph} \to \infty$ at $T = 0$. Thus, despite the qualitative overall agreement between our results and the conventional ones, there *is* need for reconciliation of the discrepancy at $T = 0$. We intend to reexamine this issue in the future.

We also note that a theory of *energy* loss (transfer) from a heated electron fluid to the lattice via the electron–phonon interaction can also be formulated in similar fashion, although this turns out to be even subtler than the analysis of momentum loss given here. This mechanism will also be treated in a separate publication.

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